JOURNAL OF APPROXIMATION THEORY 43, 299-301 (1985)

Chebyshev Approximation by Products

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Communicated by G. Meinardus

Received September 20, 1982

Characterization and uniqueness of minimax approximation by the product PQ of two finite dimensional subspaces P and Q is studied. Some approximants may have no standard characterization since PQ may not be a sun, but interior points do have the standard linear characterization.

The problem of Chebyshev best approximation by products of finite dimensional linear subspaces has been raised by several people. Boehm [1], in work prior to 1964, studied existence of a best approximation. Cheney [4, p. 109], in 1965, raised the question of whether a theory similar to that for generalized rational functions could be obtained. Gislason [6] in 1970 corrected Boehm and promised work on characterization of best approximations. In this paper we study characterization and uniqueness of best approximations.

The simplest non-trivial family of products is the set $\mathscr{G}$ of all functions of the form

$$(ax+b)(cx+d),$$

(all coefficients real). Let us consider approximation of $f(x) = 2x^2 + 1$ on $[-1, 1]$. We claim that 2 is a best approximation. As the deviation $f-2$ has extrema on $[-1, 1]$ at and only at $\{-1, 0, 1\}$, a better approximation p would satisfy

$$p(-1) > 2$$
$$p(0) < 2$$
$$p(1) > 2.$$

If p had zeros in $[-1, 1]$, we would have $\|f-p\| \geqq 1$ since $f \geqq 1$, so such p cannot be better. If p had no zeros in $[-1, 1]$, p would not be a product. Hence better p does not exist and 2 is best. By drawing a diagram it is seen

0021-9045/85 $3.00

that $\|f-3x^2\|=1$ (indeed, this is true for other multiples of x^2), so f does not have a unique best approximation. We have

$$(f-3x^2)(2-3x^2)>0$$

on the (only) extremum of $f-3x^2$ in $[-1, 1]$, namely 0. It follows that the Kolmogorov characterization of best approximations (= extremum characterizable of the author [5, p. 375]) does not hold. It follows that $\mathscr{G}$ does not have the closed-sign property [5, p. 375] (= regularity of Brosowski [3]), hence $\mathscr{G}$ is not a sun [2, p. 262]. It follows from results of the author [5, pp. 377–378] that locally best approximations can exist which are not globally best—but this can be verified directly by considering approximation of $f_k(x)=2x^2+1+1/k$. Also in the example the set of best approximations is not connected (compare Braess [2, p. 272]).

Nevertheless, it is possible to obtain a characterization and uniqueness result for *some* elements of a family PQ of products. Let R be a linear space containing all elements pq of PQ.

DEFINITION. pq is an *interior point* if all elements in a neighbourhood of pq in R are also in PQ.

THEOREM 1. *Let pq be an interior point. pq is best in PQ if and only if pq is best in R.*

Proof. As $PQ \subset R$, sufficiency is automatic. For necessity, observe that the linear proof of necessity says that if pq is not best, there is a better element in *any* neighbourhood of pq.

We can thus use the standard linear theory. In particular we have for approximation by $\mathscr{G}$ that R is the set of polynomials of degree 2, hence

THEOREM 2. *Let $g(x)=(ax+b)(cx+d)$ be an interior point. It is best to f if and only if $f-g$ alternates three times. It is unique if best.*

By the quadratic formula and continuity it is clear that all polynomials of degree 2 with two distinct zeros are interior points. A polynomial with exactly one real simple root is an interior point since nearby p have a real root nearby, hence no imaginary root. Conversely, constants and polynomials with a double zero are not interior points since they can be perturbed into a polynomial which is not a product.

It should be noted that Theorem 1 applies to mean (L_p) approximation as well.

Approximation by products of generalized rational functions (with denominators >0) is also of interest. The analogue of interior point and Theorem 1 is obvious.

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